

Charged Rotating Dilaton Black Strings

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In this paper we, first, present a class of charged rotating solutions in four-dimensional Einstein-Maxwell-dilaton gravity with zero and Liouville-type potentials. We find that these solutions can present a black hole/string with two regular horizons, an extreme black hole or a naked singularity provided the parameters of the solutions are chosen suitable. We also compute the conserved and thermodynamic quantities, and show that they satisfy the first law of thermodynamics. Second, we obtain the $(n+1)$ -dimensional rotating solutions in Einstein-dilaton gravity with Liouville-type potential. We find that these solutions can present black branes, naked singularities or spacetimes with cosmological horizon if one chooses the parameters of the solutions correctly. Again, we find that the thermodynamic quantities of these solutions satisfy the first law of thermodynamics.

I. INTRODUCTION

The Einstein equation without a cosmological constant has asymptotically flat black hole solutions with event horizon being a positive constant curvature n -sphere. However, for the Einstein equation with positive or negative cosmological constant [1, 2] or Einstein-Gauss-Bonnet equation with or without cosmological constant [3], one can have also asymptotically anti de Sitter (AdS) or de Sitter (dS) black hole solutions with horizons being zero or negative constant curvature hypersurfaces. These black hole solutions, whose horizons are not n -sphere, are often referred to as the topological black holes in the literature. Various properties of these topological black holes have been investigated in recent years. For example, the thermodynamics of rotating charged solutions of the Einstein-Maxwell equation with a negative cosmological constant with zero curvature horizons in various dimensions have been

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studied in Ref. [4], while the thermodynamics of these kind of solutions in Gauss-Bonnet gravity have been investigated in [5].

All of the above mentioned black holes are the solutions of field equations in the presence of a long-range gravitational tensor field $g_{\mu\nu}$ and a long-range electromagnetic vector fields A_μ . It is natural then to suppose the existence of a long-range scalar field too. This leads us to the scalar-tensor theories of gravity, where, there exist one or several long-range scalar fields. Scalar-tensor theories are not new, and it was pioneered by Brans and Dicke [6], who sought to incorporate Mach's principle into gravity. Also in the context of string theory, the action of gravity is given by the Einstein action along with a scalar dilaton field which is non minimally coupled to the gravity [7]. The action of $(n + 1)$ -dimensional dilaton Einstein-Maxwell gravity with one scalar field Φ and potential $V(\Phi)$ can be written as [8]

$$I_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left(\mathcal{R} - \frac{4}{n-1} (\nabla\Phi)^2 - V(\Phi) - e^{-4\alpha\Phi/(n-1)} F_{\mu\nu} F^{\mu\nu} \right) + \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-\gamma} \Theta(\gamma), \quad (1)$$

where \mathcal{R} is the Ricci scalar, α is a constant determining the strength of coupling of the scalar and electromagnetic field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field and A_μ is the vector potential. The last term in Eq. (1) is the Gibbons-Hawking boundary term. The manifold \mathcal{M} has metric $g_{\mu\nu}$ and covariant derivative ∇_μ . Θ is the trace of the extrinsic curvature $\Theta^{\mu\nu}$ of any boundary(ies) $\partial\mathcal{M}$ of the manifold \mathcal{M} , with induced metric(s) γ_{ij} . Exact charged dilaton black hole solutions of the action (1) in the absence of a dilaton potential [$V(\Phi) = 0$] have been constructed by many authors [9, 10]. The dilaton changes the casual structure of the spacetime and leads to curvature singularities at finite radii. In the presence of Liouville-type potential [$V(\Phi) = 2\Lambda \exp(2\beta\Phi)$], static charged black hole solutions have also been discovered with a positive constant curvature event horizons and zero or negative constant curvature horizons [8, 11]. Recently, the properties of these black hole solutions which are not asymptotically AdS or dS, have been studied [12].

These exact solutions are all static. Recently, One of us has constructed two classes of magnetic rotating solutions in four-dimensional Einstein-Maxwell-dilaton gravity with Liouville-type potential [13]. These solutions are not black holes, and present spacetimes with conic singularities. Till now, charged rotating dilaton black hole solutions for an arbitrary coupling constant has not been constructed. Indeed, exact rotating black hole solutions have been obtained only for some limited values of the coupling constant [14]. For general

dilaton coupling, the properties of charged dilaton black holes only with infinitesimally small angular momentum [15] or small charge [16] have been investigated. Our aim here is to construct exact rotating charged dilaton black holes for an arbitrary value of coupling constant and investigate their properties.

The outline of our paper is as follows. In Sec II, we obtain the four-dimensional charged rotating dilaton black holes/strings which are not asymptotically flat, AdS or dS, and show that the thermodynamic quantities of these black strings satisfy the first law of thermodynamics. Section III is devoted to a brief review of the general formalism of calculating the conserved quantities, and investigation of the first law of thermodynamics for the charged rotating black string. In Sec. IV, we construct the $(n + 1)$ -dimensional rotating dilaton black branes, and investigate their properties. We finish our paper with some concluding remarks.

II. ROTATING CHARGED DILATON BLACK STRINGS

The field equation of $(n + 1)$ -dimensional Einstein-Maxwell-dilaton gravity in the presence of one scalar field Φ with the potential $V(\Phi)$ can be written as:

$$\partial_\mu [\sqrt{-g} e^{-4\alpha\Phi/(n-1)} F^{\mu\nu}] = 0, \quad (2)$$

$$\mathcal{R}_{\mu\nu} = \frac{4}{n-1} \left(\nabla_\mu \Phi \nabla_\nu \Phi + \frac{1}{4} V g_{\mu\nu} \right) + 2e^{-4\alpha\Phi/(n-1)} \left(F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{2(n-1)} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu} \right), \quad (3)$$

$$\nabla^2 \Phi = \frac{n-1}{8} \frac{\partial V}{\partial \Phi} - \frac{\alpha}{2} e^{-4\alpha\Phi/(n-1)} F_{\rho\sigma} F^{\rho\sigma}. \quad (4)$$

In this section we want to obtain the four-dimensional charged rotating black hole solutions of the field equations (2)-(4) with cylindrical or toroidal horizons. The metric of such a spacetime with cylindrical symmetry can be written as

$$ds^2 = -f(r) (\Xi dt - a d\varphi)^2 + r^2 R^2(r) \left(\frac{a}{l^2} dt - \Xi d\varphi \right)^2 + \frac{dr^2}{f(r)} + \frac{r^2}{l^2} R^2(r) dz^2, \quad (5)$$

$$\Xi^2 = 1 + \frac{a^2}{l^2},$$

where the constants a and l have dimensions of length and as we will see later, a is the rotation parameter and l is related to the cosmological constant Λ for the case of Liouville-type potential with constant Φ . In metric (5), the ranges of the time and radial coordinates are $-\infty < t < \infty$, $0 \leq r < \infty$. The topology of the two dimensional space, $t = \text{constant}$ and

$r = \text{constant}$, can be (i) $S^1 \times S^1$ the flat torus T^2 model, with $0 \leq \varphi < 2\pi$, $0 \leq z < 2\pi l$, (ii) $R \times S^1$, the standard cylindrical symmetric model with $0 \leq \varphi < 2\pi$, $-\infty < z < \infty$, and (iii) R^2 , the infinite plane model, with $-\infty < \varphi < \infty$, $-\infty < z < \infty$ (this planar solution does not rotate).

The Maxwell equation (2) for the metric (5) is $\partial_\mu [r^2 R^2(r) \exp(-2\alpha\Phi) F^{\mu\nu}] = 0$, which shows that if one choose

$$R(r) = \exp(\alpha\Phi), \quad (6)$$

then the vector potential can be written as

$$A_\mu = -\frac{q}{r}(\Xi\delta_\mu^t - a\delta_\mu^\varphi), \quad (7)$$

where q is the charge parameter. In order to obtain the functions $\Phi(r)$ and $f(r)$, we write the field equations (3) and (4) for $a = 0$:

$$rf'' + 2f'(1 + \alpha r\Phi') - 2q^2r^{-3}e^{-2\alpha\Phi} + rV(\Phi) = 0, \quad (8)$$

$$rf'' + 2f'(1 + \alpha r\Phi') + 4f[\alpha r\Phi'' + 2\alpha\Phi' + r(1 + \alpha^2)\Phi'^2] - 2q^2r^{-3}e^{-2\alpha\Phi} + rV(\Phi) = 0, \quad (9)$$

$$f'(1 + \alpha r\Phi') + \alpha f[\rho\Phi'' + 2r\alpha\Phi'^2 + 4\Phi' + (\alpha r)^{-1}] - q^2r^{-3}e^{-2\alpha\Phi} + \frac{1}{2}rV(\Phi) = 0, \quad (10)$$

$$f\Phi'' + f'\Phi' + 2\alpha f\Phi'^2 + 2r^{-1}f\Phi' - \alpha q^2r^{-4}e^{-2\alpha\Phi} - \frac{1}{4}\frac{\partial V}{\partial\Phi} = 0, \quad (11)$$

where the “prime” denotes differentiation with respect to r . Subtracting Eq. (8) from Eq. (9) gives:

$$\alpha r\Phi'' + 2\alpha\Phi' + r(1 + \alpha^2)\Phi'^2 = 0,$$

which shows that $\Phi(r)$ can be written as:

$$\Phi(r) = \frac{\alpha}{1 + \alpha^2} \ln \left(\frac{b}{r} + c \right), \quad (12)$$

where b and c are two arbitrary constants. Using the expression (12) for $\Phi(r)$ in Eqs. (8)-(11), we find that these equations are inconsistent for $c \neq 0$. Thus, we put $c = 0$.

A. Solutions with $V(\Phi) = 0$

We begin by looking for the solutions in the absence of a potential ($V(\Phi) = 0$). In this case, it is easy to solve the field equations (8)-(11). One obtains

$$f(r) = r^{\gamma-1} \left(-C + \frac{(1 + \alpha^2)q^2}{V_0 r} \right),$$

where C is an arbitrary constant, $\gamma = 2\alpha^2/(1 + \alpha^2)$ and $V_0 = b^\gamma$. In the absence of a non-trivial dilaton ($\alpha = 0 = \gamma$), this solution does not exhibit a well-defined spacetime at infinity. Indeed, α should be greater than or equal to one. In order to study the general structure of these solutions, we first look for the curvature singularities. It is easy to show that the Kretschmann scalar $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ diverges at $r = 0$, it is finite for $r \neq 0$ and goes to zero as $r \rightarrow \infty$. Thus, there is an essential singularity located at $r = 0$. Also, it is notable to mention that the Ricci scalar is finite every where except at $r = 0$, and goes to zero as $r \rightarrow \infty$. The spacetime is asymptotically flat for $\alpha = 1$, while it is neither asymptotically flat nor (A)dS for $\alpha > 1$. This spacetime presents a naked singularity with a regular cosmological horizon at

$$r_c = \frac{(1 + \alpha^2)q^2}{CV_0},$$

provided $C > 0$, and no cosmological horizon for $C < 0$.

B. Solutions with Liouville-type potentials

Now we consider the solutions of Eqs. (8)-(11) with a Liouville-type potential $V(\Phi) = 2\Lambda \exp(2\beta\Phi)$. One may refer to Λ as the cosmological constant, since in the absence of the dilaton field Φ the action (1) reduces to the action of Einstein-Maxwell gravity with cosmological constant. The only case that we find exact solutions for an arbitrary values of Λ with $R(r)$ and $\Phi(r)$ given in Eqs. (6) and (12) is when $\beta = \alpha$. It is easy then to obtain the function $f(r)$ as

$$f(r) = r^\gamma \left(\frac{\Lambda V_0 (1 + \alpha^2)^2}{(\alpha^2 - 3)} r^{2(1-\gamma)} - \frac{m}{r} + \frac{(1 + \alpha^2)q^2}{V_0 r^2} \right). \quad (13)$$

In the absence of a non-trivial dilaton ($\alpha = 0 = \gamma$), the solution reduces to the asymptotically AdS and dS charged rotating black string for $\Lambda = -3/l^2$ and $\Lambda = 3/l^2$ respectively [2, 4]. As one can see from Eq. (13), there is no solution for $\alpha = \sqrt{3}$ with a Liouville potential ($\Lambda \neq 0$). In order to investigate the casual structure of the spacetime, we consider it for different ranges of α separately.

For $\alpha > \sqrt{3}$, as r goes to infinity the dominant term is the second term, and therefore the spacetime has a cosmological horizon for positive values of the mass parameter, despite the sign of the cosmological constant Λ .

For $\alpha < \sqrt{3}$ and large values of r , the dominant term is the first term, and therefore there exist a cosmological horizon for $\Lambda > 0$, while there is no cosmological horizons if $\Lambda < 0$. Indeed, in the latter case ($\alpha < \sqrt{3}$ and $\Lambda < 0$) the spacetimes associated with the solution (13) exhibit a variety of possible casual structures depending on the values of the metric parameters α , m , q , and Λ . One can obtain the casual structure by finding the roots of $f(r) = 0$. Unfortunately, because of the nature of the exponents in (13), it is not possible to find explicitly the location of horizons for an arbitrary value of α . But, we can obtain some information by considering the temperature of the horizons.

One can obtain the temperature and angular velocity of the horizon by analytic continuation of the metric. The analytical continuation of the Lorentzian metric by $t \rightarrow i\tau$ and $a \rightarrow ia$ yields the Euclidean section, whose regularity at $r = r_h$ requires that we should identify $\tau \sim \tau + \beta_h$ and $\varphi \sim \varphi + i\beta_h\Omega_h$, where β_h and Ω_h are the inverse Hawking temperature and the angular velocity of the horizon. It is a matter of calculation to show that

$$T_h = \frac{f'(r_h)}{4\pi\Xi} = \frac{r_h^{\gamma-3}}{4\pi\Xi V_0(1+\alpha^2)} [(3-\alpha^2)V_0 m r_h - 4(1+\alpha^2)q^2] \\ = -\frac{(1+\alpha^2)r_h^{\gamma-3}}{4\pi\Xi V_0} [\Lambda V_0^2 r_h^{4-2\gamma} + q^2], \quad (14)$$

$$\Omega_h = \frac{a}{\Xi l^2}. \quad (15)$$

Equation (14) shows that the temperature is negative for the two cases of (i) $\alpha > \sqrt{3}$ despite the sign of Λ , and (ii) positive Λ despite the value of α . As we argued above in these two cases we encounter with cosmological horizons, and therefore the cosmological horizons have negative temperature. Numerical calculations shows that the temperature of the event horizon goes to zero as the black hole approaches the extreme case. Thus, one can see from Eq. (14) that there exist extreme black holes only for negative Λ and $\alpha < \sqrt{3}$, if $r_h = 4(1+\alpha^2)q^2/[m_{crit}(3-\alpha^2)V_0]$, where m_{crit} is the mass of extreme black hole. If one substitutes this r_h into the equation $f(r) = 0$, then one obtains the condition for extreme black string as:

$$m_{crit} = \frac{4(1+\alpha^2)q^2}{V_0(3-\alpha^2)} \left(-\frac{\Lambda V_0^2}{q^2} \right)^{(1+\alpha^2)/4}. \quad (16)$$

Indeed, the metric of Eqs. (5) and (13) has two inner and outer horizons located at r_- and r_+ , provided the mass parameter m is greater than m_{crit} . We will have an extreme black string in the case of $m = m_{crit}$, and a naked singularity if $m < m_{crit}$. Note that Eq. (16) reduces to the critical mass obtained in Ref. [4] in the absence of dilaton field.

Since the area law of entropy is universal, and applies to all kinds of black holes and black strings in Einstein gravity, the entropy per unit length of the string is

$$\mathcal{S} = \frac{\pi \Xi V_0 r_h^{2(1+\alpha^2)^{-1}}}{2l}, \quad (17)$$

where r_h is the horizon radius. One may note that the entropy of the extreme black hole is $\pi \Xi q / (2l \sqrt{-\Lambda})$, which is independent of the coupling constant α .

Finally, it is worthwhile to mention about the asymptotic behavior of these spacetimes. The asymptotic form of the metric given by Eqs. (5) and (13) for the nonrotating case with no cosmological horizon ($\alpha < \sqrt{3}$ and $\Lambda < 0$) is:

$$ds^2 = -\frac{\Lambda V_0 (1 + \alpha^2)^2}{2(\alpha^2 - 3)} r^{2/(1+\alpha^2)} dt^2 + \frac{(\alpha^2 - 3)}{\Lambda V_0 (1 + \alpha^2)^2} r^{-2/(1+\alpha^2)} dr^2 + r^{2/(1+\alpha^2)} \left(d\varphi^2 + \frac{dz^2}{l^2} \right).$$

Note that g_{tt} , for example, goes to infinity as $r \rightarrow \infty$, but with a rate slower than that of AdS spacetimes. Indeed, the form of the Ricci and Kretschmann (\mathcal{K}) scalars for large values of r are:

$$\begin{aligned} \mathcal{R} &= \frac{6(\alpha^2 - 2)V_0}{(\alpha^2 - 3)} \Lambda r^{-2\alpha^2/(1+\alpha^2)}, \\ \mathcal{K} &= \frac{12(\alpha^4 - 2\alpha^2 + 2)V_0^2}{(\alpha^2 - 3)^2} \Lambda^2 r^{-4\alpha^2/(1+\alpha^2)}. \end{aligned}$$

As one may note, these quantities go to zero as $r \rightarrow \infty$, but with a slower rate than those of asymptotically flat spacetimes and do not approach nonzero constants as in the case of asymptotically AdS spacetimes.

III. THE CONSERVED QUANTITIES AND FIRST LAW OF THERMODYNAMICS

The conserved charges of the string can be calculated through the use of the subtraction method of Brown and York [17]. Such a procedure causes the resulting physical quantities to depend on the choice of reference background. For asymptotically (A)dS solutions, the way that one deals with these divergences is through the use of counterterm method inspired by (A)dS/CFT correspondence [18]. However, in the presence of a non-trivial dilaton field, the spacetime may not behave as either dS ($\Lambda > 0$) or AdS ($\Lambda < 0$). In fact, it has been shown that with the exception of a pure cosmological constant potential, where $\beta = 0$, no AdS or

dS static spherically symmetric solution exist for Liouville-type potential [19]. But, as in the case of asymptotically AdS spacetimes, according to the domain-wall/QFT (quantum field theory) correspondence [20], there may be a suitable counterterm for the stress energy tensor which removes the divergences. In this paper, we deal with the spacetimes with zero curvature boundary [$R_{abcd}(\gamma) = 0$], and therefore the counterterm for the stress energy tensor should be proportional to γ^{ab} . Thus, the finite stress-energy tensor in $(n + 1)$ -dimensional Einstein-dilaton gravity with Liouville-type potential may be written as

$$T^{ab} = \frac{1}{8\pi} \left[\Theta^{ab} - \Theta \gamma^{ab} + \frac{n-1}{l_{\text{eff}}} \gamma^{ab} \right], \quad (18)$$

where l_{eff} is given by

$$l_{\text{eff}}^2 = \frac{(n-1)^3 \beta^2 - 4n(n-1)}{8\Lambda} e^{-2\beta\Phi}. \quad (19)$$

As β goes to zero, the effective l_{eff} of Eq. (19) reduces to $l = n(n-1)/2\Lambda$ of the (A)dS spacetimes. The first two terms in Eq. (18) is the variation of the action (1) with respect to γ^{ab} , and the last term is the counterterm which removes the divergences. One may note that the counterterm has the same form as in the case of asymptotically AdS solutions with zero curvature boundary, where l is replaced by l_{eff} . To compute the conserved charges of the spacetime, one should choose a spacelike surface \mathcal{B} in $\partial\mathcal{M}$ with metric σ_{ij} , and write the boundary metric in ADM form:

$$\gamma_{ab} dx^a dx^a = -N^2 dt^2 + \sigma_{ij} (d\varphi^i + V^i dt) (d\varphi^j + V^j dt),$$

where the coordinates φ^i are the angular variables parameterizing the hypersurface of constant r around the origin, and N and V^i are the lapse and shift functions respectively. When there is a Killing vector field ξ on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. (18) can be written as

$$\mathcal{Q}(\xi) = \int_{\mathcal{B}} d^{n-1}\varphi \sqrt{\sigma} T_{ab} n^a \xi^b, \quad (20)$$

where σ is the determinant of the metric σ_{ij} , ξ and n^a are the Killing vector field and the unit normal vector on the boundary \mathcal{B} . For boundaries with timelike ($\xi = \partial/\partial t$) and rotational ($\varsigma = \partial/\partial\varphi$) Killing vector fields, one obtains the quasilocal mass and angular momentum

$$M = \int_{\mathcal{B}} d^{n-1}\varphi \sqrt{\sigma} T_{ab} n^a \xi^b, \quad (21)$$

$$J = \int_{\mathcal{B}} d^{n-1}\varphi \sqrt{\sigma} T_{ab} n^a \varsigma^b, \quad (22)$$

provided the surface \mathcal{B} contains the orbits of ς . These quantities are, respectively, the conserved mass, angular and linear momenta of the system enclosed by the boundary \mathcal{B} . Note that they will both be dependent on the location of the boundary \mathcal{B} in the spacetime, although each is independent of the particular choice of foliation \mathcal{B} within the surface $\partial\mathcal{M}$.

The mass and angular momentum per unit length of the string when the boundary \mathcal{B} goes to infinity can be calculated through the use of Eqs. (21) and (22),

$$\mathcal{M} = \frac{V_0}{8l} \left(\frac{(3 - \alpha^2)\Xi^2 + \alpha^2 - 1}{(1 + \alpha^2)} \right) m, \quad \mathcal{J} = \frac{(3 - \alpha^2)V_0}{8l(1 + \alpha^2)} \Xi m a. \quad (23)$$

For $a = 0$ ($\Xi = 1$), the angular momentum per unit length vanishes, and therefore a is the rotational parameter of the spacetime. Note that Eq. (23) is valid only for $\alpha < \sqrt{3}$, which the spacetime has no cosmological horizons. Of course, one may note that these conserved charges reduce to the conserved charges of the rotating black string obtained in Ref. in [4] as $\alpha \rightarrow 0$.

Next, we calculate the electric charge of the solutions. To determine the electric field we should consider the projections of the electromagnetic field tensor on special hypersurfaces. The normal to such hypersurfaces for the spacetimes with a longitudinal magnetic field is

$$u^0 = \frac{1}{N}, \quad u^r = 0, \quad u^i = -\frac{V^i}{N},$$

and the electric field is $E^\mu = g^{\mu\rho} \exp(-2\alpha\Phi) F_{\rho\nu} u^\nu$. Then the electric charge per unit length \mathcal{Q} can be found by calculating the flux of the electric field at infinity, yielding

$$\mathcal{Q} = \frac{\Xi q}{2l}. \quad (24)$$

The electric potential U , measured at infinity with respect to the horizon, is defined by [21]

$$U = A_\mu \chi^\mu|_{r \rightarrow \infty} - A_\mu \chi^\mu|_{r=r_+},$$

where $\chi = \partial_t + \Omega \partial_\varphi$ is the null generators of the event horizon. One obtains

$$U = \frac{q}{\Xi r_+}. \quad (25)$$

Finally, we consider the first law of thermodynamics for the black string. Although it is difficult to obtain the mass \mathcal{M} as a function of the extensive quantities \mathcal{S} , \mathcal{J} and \mathcal{Q} for an arbitrary values of α , but one can show numerically that the intensive thermodynamic quantities, T , Ω and U calculated above satisfy the first law of thermodynamics,

$$dM = Td\mathcal{S} + \Omega d\mathcal{J} + U d\mathcal{Q}. \quad (26)$$

IV. THE ROTATING BLACK BRANES IN VARIOUS DIMENSIONS

In this section we look for the rotating uncharged solutions of field equations (2)-(4) in $(n+1)$ dimensions with Liouville-type potential. we first obtain the static solution and then generalize it to the case of rotating solution with all the rotation parameters.

A. Static Solutions

The metric of a static $(n+1)$ -dimensional spacetime with an $(n-1)$ -dimensional flat submanifold dX^2 can be written as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{l^2}e^{2\beta\Phi}dX^2. \quad (27)$$

The field equations (2)-(4) for the above metric become

$$\frac{r}{n-1}f'' + f'(1 + \beta r\Phi') + \frac{4\Lambda r}{(n-1)^2}e^{2\beta\Phi} = 0, \quad (28)$$

$$\begin{aligned} \frac{r}{n-1}f'' + f'(1 + \beta r\Phi') + \frac{4\Lambda r}{(n-1)^2}e^{2\beta\Phi} \\ + 2(n-2)rf \left[\beta r\Phi'' + 2\beta\Phi' + r \left(\beta^2 + \frac{4}{(n-1)^2} \right) \Phi'^2 \right] = 0, \end{aligned} \quad (29)$$

$$f'(1 + \beta r\Phi') + \beta f[r\Phi'' + (n-1)(2\Phi' + \beta r\Phi'^2) + (n-2)(\beta r)^{-1}] + \frac{2\Lambda r}{n-1}e^{2\beta\Phi} = 0, \quad (30)$$

$$f\Phi'' + f'\Phi' + (n-1) \left[\beta f\Phi'^2 + r^{-1}f\Phi' - \frac{1}{2}\beta\Lambda e^{2\beta\Phi} \right] = 0. \quad (31)$$

Subtracting Eq. (28) from Eq. (29) gives

$$\beta r\Phi'' + 2\beta\Phi' + r \left(\beta^2 + \frac{4}{(n-1)^2} \right) \Phi'^2 = 0,$$

which indicates that $\Phi(r)$ can be written as:

$$\Phi(r) = \frac{(n-1)^2\beta}{4 + (n-1)^2\beta^2} \ln \left(\frac{c}{r} + d \right), \quad (32)$$

where c and d are two arbitrary constants. Substituting $\Phi(r)$ of Eq. (32) into the field equations (28)-(31), one finds that they are consistent only for $d = 0$. Putting $d = 0$, then $f(r)$ can be written as

$$\begin{aligned} f(r) &= \frac{8\Lambda V_0}{(n-1)^3\beta^2 - 4n(n-1)} r^{2\Gamma} - m r^{1-(n-1)\Gamma}, \\ V_0 &= \Gamma^{-2}c^{2(1-\Gamma)}, \quad \Gamma = 4\{(n-1)^2\beta^2 + 4\}^{-1}. \end{aligned} \quad (33)$$

One may note that there is no solution for $(n-1)^2\beta^2 - 4n = 0$. It is worthwhile to note that the Ricci scalar goes to zero as $r \rightarrow \infty$.

The Kretschmann scalar diverges at $r = 0$, it is finite for $r \neq 0$, and goes to zero as $r \rightarrow \infty$. Thus, there is an essential singularity located at $r = 0$. Again, the spacetime is neither asymptotically flat nor (A)dS, but has a regular event horizon for negative Λ at:

$$r_h = \left\{ \frac{[(n-1)^2\beta^2 - 4n]m}{\Lambda V_0} \right\}^{1/[(n-1)\Gamma-1]}, \quad (34)$$

provided $(n-1)^2\beta^2 - 4n < 0$. If $(n-1)^2\beta^2 - 4n > 0$, then the spacetime has a cosmological horizon with radius given in Eq. (34) for positive values of Λ . The Hawking temperature of the event or cosmological horizon is:

$$T_h = -\frac{\Lambda \Gamma V_0}{2\pi \Xi} r_h^{2\Gamma-1},$$

which is positive for event horizon ($\Lambda < 0$) and negative for cosmological horizon ($\Lambda > 0$).

B. Rotating solutions with all the rotation parameters

The rotation group in $(n+1)$ -dimensions is $SO(n)$ and therefore the number of independent rotation parameters for a localized object is equal to the number of Casimir operators, which is $[n/2] \equiv k$, where $[n/2]$ is the integer part of $n/2$. The generalization of the metric (27) with all rotation parameters is

$$\begin{aligned} ds^2 = & -f(r) \left(\Xi dt - \sum_{i=1}^k a_i d\phi_i \right)^2 + \frac{r^2}{l^4} e^{2\beta\Phi} \sum_{i=1}^k (a_i dt - \Xi l^2 d\phi_i)^2 \\ & - \frac{r^2}{l^2} e^{2\beta\Phi} \sum_{i=1}^k (a_i d\phi_j - a_j d\phi_i)^2 + \frac{dr^2}{f(r)} + \frac{r^2}{l^2} e^{2\beta\Phi} dX^2, \\ \Xi^2 = & 1 + \sum_{i=1}^k \frac{a_i^2}{l^2}, \end{aligned} \quad (35)$$

where a_i 's are k rotation parameters, $f(r)$ is given in Eq. (33), and dX^2 is now the Euclidean metric on the $(n-k-1)$ -dimensional submanifold with volume V_{n-k-1} .

The conserved mass and angular momentum per unit volume V_{n-k-1} of the solution calculated on the boundary \mathcal{B} at infinity can be calculated through the use of Eqs. (21) and (22),

$$\mathcal{M} = (2\pi)^k \frac{\Gamma^n V_0^{(n-1)/2}}{16\pi l^{n-k-1}} \left[n-1 + \left(n - \frac{(n-1)^2\beta^2}{4} \right) (\Xi^2 - 1) \right] m, \quad (36)$$

$$\mathcal{J}_i = (2\pi)^k \frac{\Gamma^n V_0^{(n-1)/2}}{16\pi l^{n-k-1}} \left(n - \frac{(n-1)^2 \beta^2}{4} \right) \Xi m a_i. \quad (37)$$

The entropy per unit volume V_{n-k-1} of the black brane is

$$\mathcal{S} = (2\pi)^k \frac{\Xi (\Gamma^2 V_0)^{(n-1)/2} r_h^{(n-1)\Gamma}}{4l^{n-k-1}}, \quad (38)$$

where r_h is the horizon radius. Again, it is a matter of calculation to show that the thermodynamic quantities calculated in this section satisfy the first law of thermodynamics,

$$d\mathcal{M} = Td\mathcal{S} + \sum_{i=1}^k \Omega_i d\mathcal{J}_i, \quad (39)$$

where $\Omega_i = (\Xi l^2)^{-1} a_i$ is the i th component of angular velocity of the horizon.

V. CLOSING REMARKS

Till now, no explicit rotating charged black hole solutions have been found except for some dilaton coupling such as $\alpha = \sqrt{3}$ [14] or $\alpha = 1$ when the string three-form H_{abc} is included [22]. For general dilaton coupling, the properties of charged dilaton black holes have been investigated only for rotating solutions with infinitesimally small angular momentum [15] or small charge [16]. In this paper we obtained a class of charged rotating black hole solutions with zero and Liouville-type potentials. We found that these solutions are neither asymptotically flat nor (A)dS. In the case of $V(\Phi) = 0$, the solution (which includes the string theoretical case, $\alpha = 1$) presents a black string with a regular event horizon, provided the charge parameter does not vanish. This solution has not inner horizons, and is acceptable only for $\alpha \geq 1$. Thus, it has not a counterpart for Einstein gravity without dilaton ($\alpha = 0$).

In the presence of Liouville-type potential, we obtained exact solutions provided $\beta = \alpha \neq \sqrt{3}$. These solutions reduce to the charged rotating black string of [2, 4]. We found that these solutions have a cosmological horizon for (i) $\alpha > \sqrt{3}$ despite the sign of Λ , and (ii) positive values of Λ , despite the magnitude of α . For $\alpha < \sqrt{3}$, the solutions present black strings with outer and inner horizons if $m > m_{crit}$, an extreme black hole if $m = m_{crit}$, and a naked singularity if $m < m_{crit}$. The Hawking temperature of all the above horizons were computed. We found that the Hawking temperature is negative for inner and cosmological horizons, and it is positive for outer horizons. We also computed the conserved and thermodynamics

quantities of the four-dimensional rotating charged black string, and found that they satisfy the first law of thermodynamics.

Next, we constructed the rotating uncharged solutions of $(n + 1)$ -dimensional dilaton gravity with all rotation parameters. If $(n - 1)^2\beta^2 - 4n < 0$, then these solutions present a black branes for $\Lambda < 0$, and a naked singularity for $\Lambda > 0$. If $(n - 1)^2\beta^2 - 4n > 0$, the solutions have a cosmological horizon for positive Λ , while they are not acceptable for negative values of Λ . Again we found that the thermodynamic quantities of the black brane solutions satisfy the first law of thermodynamics.

Note that the $(n + 1)$ -dimensional rotating solutions obtained here are uncharged. Thus, it would be interesting if one can construct rotating solutions in $(n + 1)$ dimensions in the presence of electromagnetic field. One may also attempt to generalize these kinds of solutions obtained here to the case of two-term Liouville potential.

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